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Reply to "Comment on 'Entanglement and chaos in the kicked top' "

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We reply to the preceding Comment that attempts to clarify the connection between chaos and entanglement exposed in our previous paper [Phys. Rev. E **83**, 016207 (2011)]. We present additional computations that show the argument exposed in the Comment to explain the entangling power of some regular states is not important in the present case. More fundamentally we argue that the example chosen in the Comment is not the most significant in order to understand why specific regular dynamics can entangle as efficiently as when the corresponding classical dynamics is chaotic.

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The quantum-classical correspondence is a fascinating topic. In the past decades, semiclassical tools have allowed for interpreting the dynamics and spectral statistics of quantum systems in terms of phase-space properties of the corresponding classical system [1]. Classical systems with chaotic dynamics have specific signatures in quantum systems, different from classical systems with regular dynamics. These signatures are universal and hold for generic quantum systems (there are departures from universality, but these departures are also well understood in semiclassical terms).

Whether there are such types of signatures ruling the generation of entanglement has remained inconclusive. The main reason is that entanglement is a specific quantum property with no classical counterpart. From the classical viewpoint, several classical features can quantum mechanically give rise to entanglement. Although an interaction between two particles is common to all the works that have been performed so far, the role of this interaction with regard to the dynamics of the uncoupled particles has taken many forms. Although some general trends can be expected—in particular, chaotic dynamics will tend to be correlated with high quantum entanglement—it is unlikely that strong universal statements as the ones mentioned above for spectral statistics can be formulated for entanglement.

In a previous work [2], to be labeled as KT, we investigated entanglement in the quantum kicked top as a function of the dynamics of the classical kicked top. Previous results [3,4] suggested that entanglement generation in the quantum top is correlated with chaos in the classical counterpart and that, at least in this system, entanglement could be seen as a signature of chaos. On the other hand our own works [5,6] on a more general two-particle top (that yields the standard kicked top as a limiting case) indicated that entanglement dynamics did depend on classical phase-space properties but in a specific and system-dependent manner rather than in a universal fashion. Although the two-particle top has more complex richer dynamics than the kicked top, our findings in KT were in line with the results for the two-particle top, displaying regular states with a high entangling power. Notice that several other authors (e.g., Refs. [7–9]) investigating systems other than the simple kicked top have also concluded that large entanglement can occur in nonchaotic cases.

In the accompanying Comment on KT [10], Madhok makes two points. (i) First, he discusses a specific example given in Fig. 9 of KT where, in a mixed phase-space situation, we had compared the entanglement for initial states lying in regular and chaotic regions. Madhok argues that in mixed phase space, regular states will tend to flood the chaotic sea (and gives an illustration of this effect), so it is not appropriate to attribute the dynamical evolution and the ensuing entanglement generation to regular dynamics. (ii) Second, Madhok takes a single initial coherent state and varies the coupling κ, going from regular to chaotic classical dynamics; he then shows that the time-averaged entanglement increases with κ.

Concerning point (i), we start by noting that the mixed phase-space illustration portrayed in Fig. 9 of KT is not the most compelling of our examples displaying entanglement with regular states and dynamics. For the modified kicked top, Figs. 7 and 9 of Ref. [6] show initial regular states evolving with pure regular dynamics entangling as efficiently as states evolving in a fully chaotic regime. Figures 4 and 6 of KT also illustrate entanglement in a regular regime reaching maximal entanglement (in a slower time but staying longer at maximum entanglement). From the point of view of the quantum-classical correspondence, mixed phase is notoriously more difficult to deal with than the integrable and fully chaotic limits (see, e.g., Sec 8.12 of Ref. [1]).

Although the argument given in the Comment concerning the flooding of the wave packet in the regular island onto the chaotic sea, or the blurring due to the finite size of Planck cells is cogent, it remains to be shown that the associated effects are at play. This does not seem to be the case. Indeed, recall the linear entropy $S_2(t)$ employed to quantify entanglement is given by

$$S_2(t) = \frac{1}{2} - \frac{1}{2J^2}(\langle J_x \rangle^2 + \langle J_y \rangle^2 + \langle J_z \rangle^2), \tag{1}$$

where J is the angular momentum. Although $\langle J_i \rangle^2$ should be small in order for $S_2(t)$ to be large, the behavior of the quantum averages $\langle J_i \rangle$ is also an indicator of the wave packet distribution. In the case of Fig. 9, $\langle J_x \rangle$ is particularly relevant

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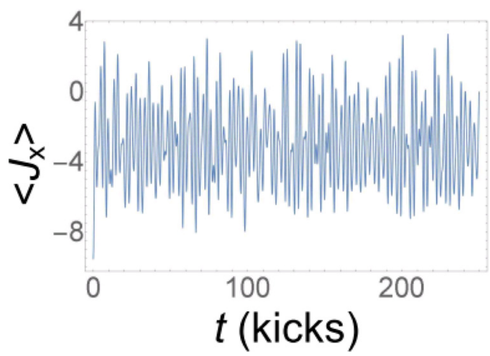

FIG. 1. (Color online) Average of the angular momentum projection $J_x(t)$ for a kicked top with $T_e = 0.95$ and $k = 5$ when the initial distribution is in the chaotic region shown in Fig. 9 of Ref. [2]. Compare with the evolution of $J_x(t)$ when the initial distribution lies in the regular region, shown in Fig. 10 of Ref. [2].

(the x axis crosses the Poincaré sphere at the center of the large regular island). As can be inferred from Figs. 9(d)–9(f), when the initial state is in a regular region, most of the classical distribution remains in the island, thereby encircling the x axis. This is reflected in $\langle J_x \rangle$, shown in Fig. 10 of KT, that remains positive and mostly confined in the interval $1 \lesssim \langle J_x \rangle \lesssim 4$, consistent with an evolution of most of the quantum wave packet around the x axis. On the other hand, when the initial state is in the chaotic sea the classical distribution spreads throughout the chaotic sea [Figs. 9(a)–9(c)]. The quantum average $\langle J_x \rangle$, not given in KT, is shown in Fig. 1 of the present Reply: It lies in the interval $-6 \lesssim \langle J_x \rangle \lesssim 1$ at the back of the sphere and hence away from the main regular region.

The comparison of $\langle J_x \rangle$ in the two cases is not consistent with the idea of substantial flooding since in that case we would expect $\langle J_x \rangle$ to display after a few kicks a similar behavior for wave packets lying initially in regions associated with classical regular and chaotic dynamics. In contrast, in the example given in Fig. 1 of the Comment, the initial state is closer to the edge of the regular island, and a non-negligible fraction of the initial distribution lies in the chaotic sea. Note that for the two-particle top, a similar mixed phase-space situation was examined [11] for higher quantum numbers ($J = 100$); no substantial flooding can be inferred from the results given in Figs. 2(a) and 2(b) of Ref. [11], which show $S_2(t)$ for states evolving in a regular island and the chaotic sea, respectively (the corresponding Poincaré surfaces of section are given in Figs. 3(a)–3(c) of Ref. [11]), and compare $S_2(t)$ with the classical counterpart (the mutual information obtained from classical distributions). The agreement between $S_2(t)$ and the classical mutual information in Ref. [11] indicates that finite $\hbar$ and edge effects are not important enough to affect, to first order in $\hbar$, the quantum-classical correspondence.

At any rate, it is important to stress that pure regular dynamics (i.e., without any possible edge effects) achieves high entanglement in the kicked top. We display here in Fig. 2(a) a case similar to Fig. 9 of KT but with a coupling k reduced from 5 to 1 and larger quantum numbers ($J = 100$). The resulting phase space is nearly entirely regular, being filled with Kolmogorov-Arnold-Moser (KAM) tori seen here as trajectories on the sphere. We highlight two of them, the separatrix (in red), and a particular trajectory on which we will launch a wave packet (in green). Notice first that chaos is negligible for $k = 1$. As usual chaos will start from the separatrix to become sizable for larger values of k as in

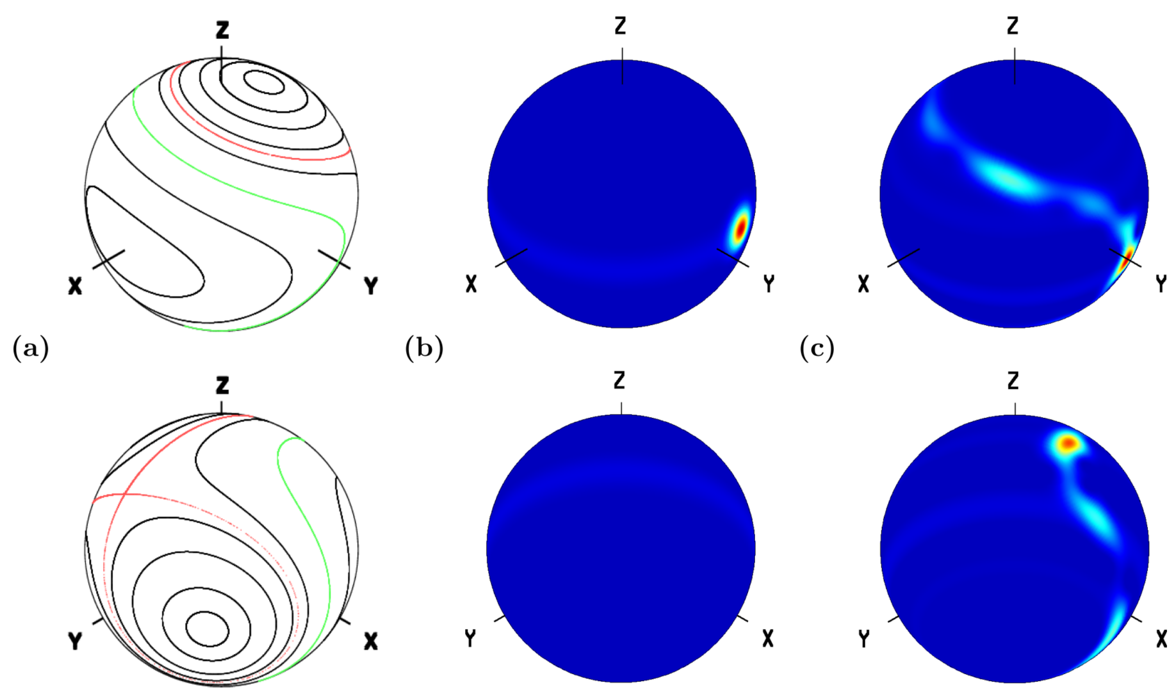

FIG. 2. (Color online) Case (a) Poincaré surface of the section (top: view of the front of the sphere and bottom: back of the sphere) for $T_e = 0.95$ as in Fig. 9 of Ref. [2], but $k = 1$. The separatrix is shown in red, and a particular KAM torus is shown in green. (b) Initial Husimi distribution, centered on the KAM torus shown in green in (a). (c) After 40 000 iterations, well above all needs: The distribution spreads over the green KAM torus without any sizeable crossing of the separatrix.

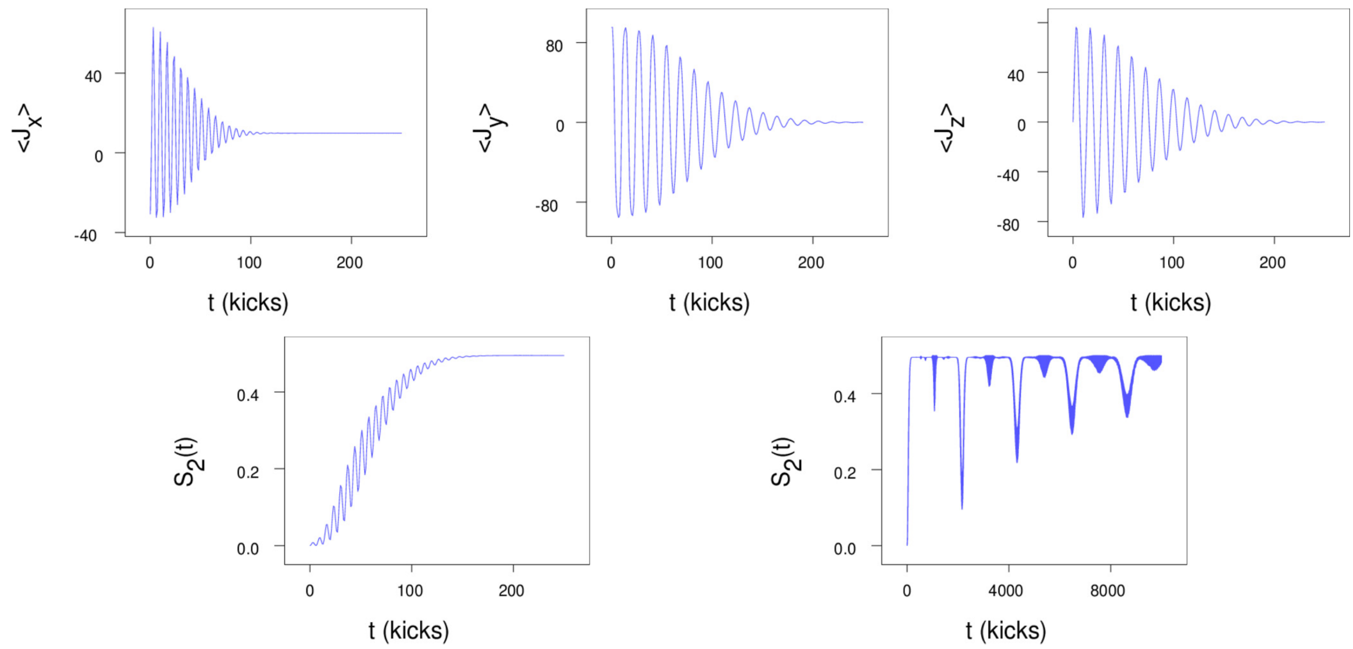

FIG. 3. (Color online) (Top) Averages of the angular momentum projections of $\vec{J}(t)$ for a kicked top with $J = 100$, $T_e = 0.95$, and $k = 1$, corresponding to Fig. 2 for short times. (Bottom) $S_2(t)$ for short and long times.

Fig. 9 of KT. As is well known [12], the KAM tori fill nearly all the phase space when $k \to 0$, and in dimension $2 + 1$ they act as barriers for classical trajectories. We then launch a Husimi packet initially centered on the chosen trajectory (green), which is away enough from the separatrix to give a negligible overlap at $t = 0$ between the exponentially decreasing tail of the distribution and the separatrix. We see that after 40 000 iterations (we have checked up to more than 10^6) the distribution remains confined between the trajectories which limit it at $t = 0$. There is no flooding across the separatrix. Semiclassically this is an obvious consequence of the containment of the classical trajectories, and we see that this remains valid for the corresponding quantum calculation.

We then display in Fig. 3 the evolution of the averages $\langle J_i \rangle$ and of the linear entropy $S_2(t)$ employed to quantify entanglement, which starts from zero to attain nearly 0.5, i.e., a fully entangled value, after a few hundred kicks.

The results immediately follow from Eq. (1). If a small wave packet is launched from the center of a regular island, it remains fixed on a point of the sphere, and the averages $\langle J_i \rangle$ are nearly equal to their center value as a point on the sphere so that the second term of Eq. (1) is equal to the first and $S_2 \sim 0$. This will be valid each time the wave packet will be of the limited (small) size allowed by the high value of J. For the evolution at hand, we see following the green line, first that $J_x(t)$ starts from a small negative value and oscillates between negative and positive values with a small positive average value. Then the wave packet spreads along the torus. The averages $\langle J_i(t) \rangle$ become roughly equal to a uniform distribution averaged on this strip. $S_2(t)$ then goes to a limiting (constant) value near 0.5, depending on the geometry of this strip. For longer times there are periodic "revivals" when the packet returns to some concentrated shape as could be supposed from the local concentrations for long times in Fig. 2. But this is increasingly unlikely so that $S_2(t)$ tends to its limiting large value. Maximal entanglement is hence achieved without flooding playing any role.

Finally, notice the similarity of this explanation with that given at the end of Sec. IV A of KT for the case regular at resonance. In this latter case there is no separatrix, phase space is homogeneously regular, but the phenomenon is the same. Simply the shape of the strip elongated along a KAM torus is different. These shape considerations seem relevant, whereas presence or absence of chaos plays no role. It seems that presence of chaos is relevant only as far as full chaos implies occupation of a strip extended to the whole sphere. But it does not give a larger $S_2(t)$ than a simple strip with minimal width so that one cannot deduce any amount of chaos from a large value of $S_2(t)$.

Concerning point (ii), we first remark that if the take-home message of this point is that "*generically regular classical dynamics will tend to be correlated with lower quantum entanglement than when the classical dynamics is chaotic*," we fully agree—this citation is precisely copied from Sec. V of KT. Actually it is known that states that are quantum chaotic (in the sense of random matrix theory) will display near maximal bipartite entanglement [13]. On the other hand, we remain skeptical that the terms "universal signature of chaos," employed without a restrictive definition, convey this meaning (in particular, these terms have been employed in the past to correlate entanglement with chaos in definite situations involving the evolution of a single initial state for a given kicked top with fixed dynamics, not by averaging over families of initial states or different kicked tops). We are also not convinced that choosing a fixed distribution on the sphere and computing the time-averaged entanglement for different kicked tops characterized by increasing values of κ as shown in Fig. 2 of the Comment is significant. At the very least, averages over the initial states must be taken (this was subsequently performed in the Comment).

But even when averaging over the sphere (of specific regions thereof), two important issues should be kept in mind. First, the functional form of the initial state will be crucial, not when the dynamics is uniform over the sphere (i.e., in the chaotic limit), but when the dynamics is regular. Indeed, choosing coherent states as initial distributions is rather arbitrary—in our modified kicked tops, it is physically more meaningful to choose slices of the sphere as initial states (as illustrated, e.g., in Fig. 3 of Ref. [11]). Averaging over the sphere employing these slices or coherent states will not give the same result when the dynamics over phase space is not uniform. Note that in order to circumvent this problem, intrinsic state-independent measures of the entangling power of unitary operators have been proposed [8]. The second issue concerns the entanglement of the eigenstates, which is intrinsic (in the sense that they are not relative to choices of initial distributions). In the chaotic regime, the eigenstates will tend to display high entanglement. In the regular regime, the structure of phase-space is highly correlated with the degree of entanglement. This was nicely illustrated in Fig. 5 of Ref. [6] in the case of the two-particle kicked top: For a fixed value of κ corresponding to regular dynamics, phase space is modified when the period of the kick and the period of the kicked particle become resonant; at resonance, many eigenstates are as highly entangled as eigenstates in the chaotic regime.

To conclude, we stay firm with our previous findings according to which entanglement is not a clear-cut signature of chaos as compared for example to the signatures of the classical dynamical regime visible in the spectral statistics of the quantum system. Notwithstanding, it would be valuable to pursue the work undertaken in the preceding Comment [10] in order to bring out whether there exist situations for which entanglement could be taken as a nonambiguous indicator of chaos.

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