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# Three-box paradox and ‘Cheshire cat grin’: the case of spin-1 atoms

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## Abstract

In this work we propose a definite theoretical implementation of the three-box paradox—a scheme in which a single quantum particle appears to be present with certainty in two separate boxes—with spin-1 atoms. We further show how our setup can give rise to a ‘Cheshire cat grin’ type of situation, in which an atom can apparently be found with certainty in one of the boxes while one of its properties (the angular momentum projection along a specifically chosen axis) appears to be in a different box. The significance of our findings is discussed relative to the status of the properties of a system obtained from weak measurements.

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(Some figures may appear in colour only in the online journal)

## 1. Introduction

What is the value of a physical property prior to a measurement or between two measurements? In the context of standard quantum mechanics the question does not make sense: trying to answer the question implies disturbing the system thereby changing the nature of the experiment. Appealing to counterfactual arguments to understand the situation rarely helps, as one is usually led to apparent paradoxes and odd behavior, such as answering the ‘which path?’ question in the well-known Wheeler delayed choice experiment [1].

The three-box thought experiment proposed by Aharonov and Vaidman [2] leads to that type of situation. The setup involves three separate boxes  $A$ ,  $B$  and  $C$  and one particle. Given suitable initial and final wavefunctions (consisting of a cleverly chosen superposition of the particle being in one of the boxes), one wants to know in which box the particle can be found at some intermediate time. The quantum formalism seems to indicate that if a measurement

could be made to detect the particle in, say box  $A$ , while allowing the system to reach the final state, then the particle would be found in box  $A$  with certainty. But if box  $B$  were opened instead, then the particle would also be found there with certainty, although it is of course impossible to find with certainty a single particle in two different boxes. A closely related setup involves a particle and one of its properties, e.g., the spin: the particle appears to have taken with certainty one of two given paths (on the grounds that if the particle presence along the other path could be measured—while allowing the system to reach the final state—the probability of finding it there would be zero) while the spin appears to have taken the other path with certainty. This setup, which has recently received increased interest [3–6], was given [7] the suggestive name of ‘Cheshire cat’ since the grin (the property) appears to be separated from the cat (the particle).

These apparent paradoxes are based on counterfactual inferences—opening a box disturbs the system which does not reach the final state—so that, irrespective of whether trying to answer the ‘which path?’ or ‘which box?’ question is or is not legitimate, the effects cannot be observed. However, counterfactual reasoning can to some extent be bypassed by employing a scheme known as weak measurement (WM) [8]. In contrast to the usual projective measurements, a WM consists of a weak unitary interaction coupling the system with a meter. The system, largely unperturbed, reaches the final state. The meter wavefunction correlated with that final state indicates the *weak value* of the weakly measured system observable. WM thus appears as a tool to non-invasively open the boxes and ascertain what is happening inside. Indeed WMs have been employed experimentally in dozens of works, essentially with optical setups, confirming the theoretical predictions (though the interpretation of WM remains a controversial topic). In particular an optical version of the three-box experiment was realized [9] in a modified Mach–Zehnder style interferometer.

In this work, we propose a theoretical implementation of the three-box setup and of a ‘Cheshire cat grin’ scheme employing spin-1 particles. We will have more specifically in mind spin-1 atoms, that have been extensively manipulated in atomic interferometry experiments [10], including the use of Stern–Gerlach types of devices which are an essential tool in the setups presented below. One motivation is that while an optical three-box experiment can be explained in classical terms (based on classical interference effects), this is of course not the case of experiments performed with massive particles. Moreover, the theoretical account employing spin-1 atoms involves the explicit wavepacket dynamical evolution, contrary to the original idealized three-box thought experiment. Employing a well-defined physical system dispels in our view many ambiguities that have given rise to controversies (eg [11, 12, 14]) discussed in relation with the ideal three-box paradox.

This paper is organized as follows. The original three-box paradoxes, either based on counterfactual arguments or on WMs are recalled in section 2. The implementation of the thought experiment with spin-1 particles is described in section 3. A Cheshire cat grin type of scheme, based on the setup described in section 3, is developed in section 4, with the ‘grin’ taken to be the spin projection on a chosen axis. The results and their significance along with some remarks in view of a possible experimental implementation with atoms of the proposed schemes is given in section 5.

## 2. The three-box thought experiment: counterfactuals and weak measurements

### 2.1. The three-box example and counterfactuals

The three-box paradox [2] is usually presented in the context of time-symmetric quantum mechanics [15] as illustrating a complete description of a quantum system at any time given a

fixed initial (known as ‘pre-selected’) state and a fixed final (termed as ‘post-selected’) state. Assume we have a quantum system that can be in one of the three boxes  $A$ ,  $B$  or  $C$ . The mutually orthogonal states  $|A\rangle$ ,  $|B\rangle$  and  $|C\rangle$  label the particle as being in one of the respective boxes. Let the system be initially ( $t = t_i$ ) prepared in the state

$$|\psi_i\rangle = \frac{1}{\sqrt{3}} (|A\rangle + |B\rangle + |C\rangle) \quad (1)$$

and post-selected at  $t = t_f$  to the final state

$$|\psi_f\rangle = \frac{1}{\sqrt{3}} (|A\rangle + |B\rangle - |C\rangle). \quad (2)$$

What would happen if one of the boxes were opened at some intermediate time between  $t_i$  and  $t_f$ ? Assume box  $A$  is opened; if the particle is found there, this means the initial state has been projected to state  $|A\rangle$  through  $\Pi_A|\psi_i\rangle$  where  $\Pi_A \equiv |A\rangle\langle A|$ . If the particle is not found there, then the state after box  $A$  is opened is  $\Pi_{\bar{A}}|\psi_i\rangle$  where

$$\Pi_{\bar{A}} \equiv 1 - \Pi_A = |B\rangle\langle B| + |C\rangle\langle C| \quad (3)$$

is the complement of  $\Pi_A$ . However the transition amplitude  $\langle\psi_f|\Pi_{\bar{A}}|\psi_i\rangle$  to the final post-selected state vanishes, so the probability of not finding the particle in box  $A$ , proportional to  $|\langle\psi_f|\Pi_{\bar{A}}|\psi_i\rangle|^2$ , is zero: the conclusion is therefore that the particle *must* have been with certainty in box  $A$ .

A contradiction arises by repeating the same argument assuming now that box  $B$  is opened. If the particle is not found in box  $B$ , then the state after box  $B$  has been opened is  $\Pi_{\bar{B}}|\psi_i\rangle$  with  $\Pi_{\bar{B}} \equiv 1 - \Pi_B = |A\rangle\langle A| + |C\rangle\langle C|$ . But  $|\langle\psi_f|\Pi_{\bar{B}}|\psi_i\rangle|^2 = 0$ , so on its way from  $|\psi_i\rangle$  to  $|\psi_f\rangle$  the particle *must* have been with certainty in box  $B$ ! Of course, a single particle cannot be with certainty in two different boxes at the same time, hence the paradox.

The paradox is apparently dissolved by remarking that quantum mechanics does not allow this type of counterfactual reasoning. It does not make sense, according to the standard interpretation, to demand a ‘complete’ description of the behavior of the system between  $|\psi_i\rangle$  and  $|\psi_f\rangle$  without actually making a measurement at box  $A$  and/or box  $B$  that will disturb the system. In particular, opening the two boxes jointly will yield a single particle either in box  $A$  or box  $B$  (or none), and the quantum formalism consistently predicts  $\Pi_A\Pi_B|\psi_i\rangle = 0$ . Nevertheless, actually performing these measurements changes the nature of the experiment: the particle never reaches the final state  $|\psi_f\rangle$ , either because it is detected in the box or because post-selection is not successful if it is undetected, leaving the original question relative to the behavior of the system at an intermediate time unanswered.

## 2.2. Weak measurements

WM represents a tool that provides a certain type of answer to the question. Briefly put, the main idea consists in two steps: first a system in a pre-selected state weakly interacts unitarily with an apparatus, resulting in an entangled system–apparatus state; the interaction couples a system observable  $O$  with a dynamical variable of the apparatus. Then a standard projective measurement of a different system observable is made; one retains only the outcomes leaving the system in the chosen post-selected state. The corresponding projection leaves the apparatus wavefunction in a certain final state. Under certain conditions, basically amounting to a very weak interaction and widely overlapping meter states [16, 17], the final state is simply shifted relative to the initial state, the shift being proportional to the weak value  $\langle O \rangle_w$  of the system observable that was weakly coupled to the meter. The weak value is given by

$$\langle O \rangle_w = \frac{\langle\psi_f|O|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}. \quad (4)$$

In the present context, we see that WM allows us to obtain information on some system observable  $O$  at some intermediate time while the system evolves from  $|\psi_i\rangle$  to  $|\psi_f\rangle$ . Indeed, the weak coupling barely affects the system while the meter wavefunction picks up a phase shift that can in principle be experimentally detected. WM thus appears as a way to bypass counterfactual reasoning and access what is happening in a system between the initial and final states.

### 2.3. The three-box paradox

The analysis of the three-box paradox with WM [2, 14, 18] involves replacing the measurements (box openings) and their associated projectors  $\Pi_A, \Pi_B, \dots$  with WM and the respective weak values  $\langle \Pi_A \rangle_w, \langle \Pi_B \rangle_w, \dots$ . From the state definitions (1) and (2) it is straightforward to apply equation (4). This gives

$$\langle \Pi_A \rangle_w = 1 \quad \text{and} \quad \langle \Pi_{\bar{A}} \rangle_w = 0 \quad (5)$$

meaning that an apparatus weakly interacting with box  $A$  will move, but that a single meter that would weakly open boxes  $B$  and  $C$  jointly (see below for a definite example in the context of spin-1 particles) will not display any shift. If WMs were to be interpreted along the same line as projective measurements, the conclusion would be that on its way to  $|\psi_f\rangle$  the particle went through box  $A$ .

However we also have

$$\langle \Pi_B \rangle_w = 1 \quad \text{and} \quad \langle \Pi_{\bar{B}} \rangle_w = 0 \quad (6)$$

leading to the conclusion that on its way to  $|\psi_f\rangle$  the particle went through box  $B$ . But now, unlike the case with projective measurements discussed above, no counterfactual arguments are involved: if two apparati open weakly boxes  $A$  and  $B$  respectively, we will have jointly  $\langle \Pi_A \rangle_w = 1$  and  $\langle \Pi_B \rangle_w = 1$ . In one sense the paradox is back again, though whether with WM there is a real paradox involved or whether WM allows us to observe the superpositions typical of quantum phenomena is largely a matter of interpretation (see section 5 below).

## 3. The three-box paradox with spin-1 particles

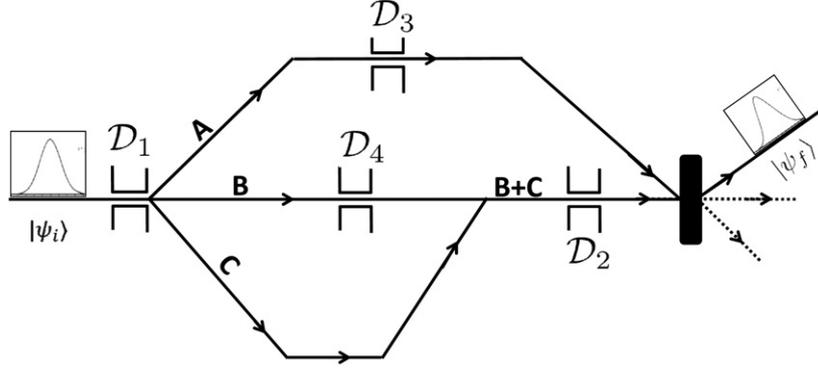
### 3.1. General remarks

The three-box paradox, as initially proposed in [2], was realized experimentally with photons [9]. As remarked by the authors of [9], the effects giving rise to the paradox have a classical optical explanation in terms of the classical interference of light. On the other hand spin-1 particles would allow us to realize experimentally the three-box paradox with matter waves. Here we will simply describe the spin-1 version of the three-box paradox without considering any experimental realization (a possible experimental realization is outlined in section 5).

Assume spin-1 particles (e.g., atoms) are prepared in the initial state

$$|\psi_i\rangle = |J = 1, m_z = 0\rangle |\xi\rangle \quad (7)$$

where  $\xi(\mathbf{r}) \equiv \langle \mathbf{r} | \xi \rangle$  is the spatial part of the wavefunction and  $|J = 1, m_z = 0\rangle \equiv |m_i\rangle$  is the  $J = 1$  spin state (we will omit explicitly denoting  $J = 1$  in the rest of the paper) with the spin projection quantized along the  $\hat{\mathbf{z}}$  axis with azimuthal number  $m_z = 0$ . We assume  $\xi(\mathbf{r})$  can be represented by a Gaussian, moving to the right on the  $\hat{\mathbf{y}}$  axis (see figure 1), and its width depending on the coherence length of the atoms; we will explicitly write the spatial part of the wavefunction only when appropriate, focusing on the sole spin part in most of the paper.



**Figure 1.** The ‘three-box paradox’ setup for spin-1 particles. Instead of three boxes, a particle prepared in an initial pre-selected state  $J = 1, M = m_i$ , enters a Stern–Gerlach type of device  $\mathcal{D}_1$  that separates the  $m_i$  state on the  $m_\alpha$  basis: the wavepacket is then divided along the three paths  $A, B$  and  $C$ . The wavepackets along  $B$  and  $C$  are recombined first, and then recombined with the wavepacket traveling along path  $A$ , at which point a projective measurement (represented by the black box) is made. The ‘apparati’  $\mathcal{D}_2, \mathcal{D}_3$  and  $\mathcal{D}_4$  interact unitarily with the system.

Assume that at  $t = 0$  the wavepacket enters a SG type of device ( $\mathcal{D}_1$  in figure 1) with an inhomogeneous magnetic field directed along the direction  $\hat{\alpha}$ . The effect of  $\mathcal{D}_1$  is to separate the wavepackets according to their associated spin projection along  $\hat{\alpha}$ . The spin part of the initial state in the  $|m_\alpha\rangle$  basis is transformed as

$$|\psi_i\rangle = |m_i\rangle|\xi\rangle \equiv \sum_{k=-1}^1 \langle m_\alpha = k | m_z = 0 \rangle |m_\alpha = k\rangle |\xi(t = 0)\rangle, \quad (8)$$

where

$$\langle m_\alpha | m_\beta \rangle \equiv d_{m_\alpha, m_\beta}^{J=1}(\beta - \alpha) \quad (9)$$

is given by the reduced Wigner rotational matrix element. For  $t > 0$ ,  $|\xi\rangle$  separates into three wavepackets each associated with a given value of  $m_\alpha$ , so that upon exiting  $\mathcal{D}_1$  the system wavefunction becomes

$$|\psi(t)\rangle = \sum_k \langle m_\alpha = k | m_z = 0 \rangle |m_\alpha = k\rangle |\xi_k(t)\rangle; \quad (10)$$

the  $|\xi_k\rangle$  can be computed by solving the Schrödinger equation inside  $\mathcal{D}_1$  [17]. Each of the three paths is taken to represent a box: box  $A$  is taken to be the  $k = +1$  path, with boxes  $B$  and  $C$  corresponding respectively to the paths  $k = 0$  and  $k = -1$ . We assume that paths  $B$  and  $C$  are recombined first and then recombined with path  $A$  (as shown in figure 1). These recombinations are assumed to take place without affecting the spin state nor the phase difference. Finally a projective measurement of the spin projection along the direction  $\hat{\phi}$  is made at time  $t_f$ . The final post-selected state is chosen to be

$$|\psi_f\rangle = |m_f\rangle|\xi(t_f)\rangle \equiv \sum_{k=-1}^1 \langle m_\alpha = k | m_\phi = +1 \rangle |m_\alpha = k\rangle |\xi(t_f)\rangle \quad (11)$$

with  $|m_f\rangle \equiv |m_\phi = +1\rangle$ .

### 3.2. Condition on path A

In order to obtain the analogue of the three-box paradox, some transition amplitudes must interfere destructively. Assume an apparatus  $\mathcal{D}_2$  is positioned at  $\mathbf{r}_2$  as indicated in figure 1.  $\mathcal{D}_2$  measures the weak value of the projector  $\Pi_{\bar{A}}$  (equation (3)) along the recombined path  $B + C$ . The projector  $\Pi_{\bar{A}}$  can be taken here to project to a Gaussian centered on  $\mathcal{D}_2$  whose width encompasses the spatial extent of the spatial part of the wavepackets,  $\Pi_{\bar{A}} = |\Gamma\rangle\langle\Gamma|$  where

$$\Gamma(\mathbf{r}) = \left(\frac{2}{\pi\Delta^2}\right)^{1/2} e^{-(\mathbf{r}-\mathbf{r}_2)^2/\Delta^2}. \quad (12)$$

The weak value  $\langle\Pi_{\bar{A}}\rangle_w$  is obtained from equation (4) keeping in mind that we are dealing with a ‘real’ system endowed with dynamics (an element that is not taken into account in the ideal three-box paradox presented in section 2). Denoting by  $t_2$  the time at which  $\mathcal{D}_2$  is triggered (and the WM made) and introducing the evolution operator  $U(t_k, t_j)$  of the spin-1 particle between  $t_j$  and  $t_k$  we have

$$\langle\Pi_{\bar{A}}\rangle_w = \frac{\langle\psi_f(t_f)|U(t_f, t_2)\Pi_{\bar{A}}U(t_2, t_i)|\psi_i\rangle}{\langle\psi_f|U(t_f, t_2)U(t_2, t_i)|\psi_i\rangle}. \quad (13)$$

Following equation (10),  $U(t_2, t_i)|\psi_i\rangle$  is of the form

$$U(t_2, t_i)|\psi_i\rangle = \langle m_\alpha = +1|m_i\rangle|m_\alpha = +1\rangle|\xi_A(t_2)\rangle + \sum_{k=-1,0} \langle m_\alpha = k|m_i\rangle|m_\alpha = k\rangle|\xi_{B+C}(t_2)\rangle \quad (14)$$

and  $\Pi_{\bar{A}}|\xi_A(t_2)\rangle$  vanishes (since there is no spatial overlap between  $|\Gamma\rangle$  and  $|\xi_A(t_2)\rangle$ ). The weak value becomes

$$\langle\Pi_{\bar{A}}\rangle_w = \frac{\langle\xi(t_f)|U(t_f, t_2)\Pi_{\bar{A}}|\xi_{B+C}(t_2)\rangle}{\langle\psi_f|U(t_f, t_2)U(t_2, t_i)|\psi_i\rangle} \left[ \sum_{k=-1,0} \langle m_\alpha = k|m_i\rangle\langle m_f|m_\alpha = k\rangle \right]. \quad (15)$$

Hence  $\hat{\phi}$  needs to be chosen (restricting  $\hat{\phi}$  to lie in the  $yz$  plane) such that the transition amplitudes  $m_\alpha = 0 \rightarrow m_\phi = +1$  and  $m_\alpha = -1 \rightarrow m_\phi = +1$  interfere destructively, viz by solving

$$\sum_{k=-1,0} \langle m_f|m_\alpha = k\rangle\langle m_\alpha = k|m_i\rangle = 0. \quad (16)$$

One class of solutions to equation (16) for a fixed value of  $\alpha$  is given by

$$\phi = 4 \arctan \left( \frac{8 \tan^3 \frac{\alpha}{4} - \frac{(3 \cos(2\alpha) + 5)^{1/2} \sec^6(\frac{\alpha}{4})}{2\sqrt{2}}}{(\tan^2 \frac{\alpha}{4} - 1)^3} \right) + 4\pi n. \quad (17)$$

Therefore provided  $\alpha$  and  $\phi$  obey equation (17), we have  $\langle\Pi_{\bar{A}}\rangle_w = 0$ : the apparatus  $\mathcal{D}_2$  will not display any change following post-selection.

Conversely if an apparatus  $\mathcal{D}_3$  is positioned at  $\mathbf{r}_3$  along path A as indicated in figure 1 the weak value of the projector  $\Pi_A$  is given by

$$\langle\Pi_A\rangle_w = \frac{\langle\psi_f(t_f)|U(t_f, t_3)\Pi_A U(t_3, 0)|\psi_i\rangle}{\langle\psi_f|U(t_f, t_3)U(t_3, t_i)|\psi_i\rangle}, \quad (18)$$

where  $t_3$  is the time at which the WM is made. Employing equations (10) and (11) and keeping in mind  $\Pi_A|\xi_k(t_3)\rangle = 0$  for  $k = B, C$  leads to

$$\langle\Pi_A\rangle_w = \frac{\langle\xi_f(t_f)|U(t_f, t_3)\Pi_A|\xi_A(t_3)\rangle\langle m_f|m_\alpha = 1\rangle\langle m_\alpha = 1|m_i\rangle}{\sum_{k=-1}^1 \langle m_f|m_\alpha = k\rangle\langle m_\alpha = k|m_i\rangle} \quad (19)$$

which simplifies given the condition (16) to

$$\langle \Pi_A \rangle_w = \langle \xi(t_f) | U(t_f, t_3) \Pi_A | \xi_A(t_3) \rangle. \quad (20)$$

Hence the meter corresponding to the apparatus  $\mathcal{D}_3$ , interacting with the spin-1 system, moves, the motion of the pointer being proportional to the system and meter wavepackets overlap. Note that in the ideal case in which the projector  $\Pi_A$  perfectly overlaps with the system wavepacket at the time of measurement, i.e.  $\Pi_A \equiv |\xi_A(t_3)\rangle\langle\xi_A(t_3)|$  we have (since  $U(t_f, t_3)|\xi_A(t_3)\rangle = |\xi(t_f)\rangle$ )

$$\langle \Pi_A \rangle_w = 1 \quad (21)$$

as in equation (5) of the ideal three-box paradox.

### 3.3. Condition on path B

By employing the same reasoning followed for path A, let us now position an apparatus  $\mathcal{D}_4$  at  $\mathbf{r}_4$  along path B as indicated in figure 1. The weak value of the projector  $\Pi_B$ , measured by  $\mathcal{D}_4$  at time  $t_4$  is given by

$$\langle \Pi_B \rangle_w = \frac{\langle \psi_f(t_f) | U(t_f, t_4) \Pi_B U(t_4, 0) | \psi_i \rangle}{\langle \psi_f | U(t_f, t_4) U(t_4, t_i) | \psi_i \rangle} \quad (22)$$

taking the form

$$\langle \Pi_B \rangle_w = \frac{\langle \xi_f(t_f) | U(t_f, t_4) \Pi_B | \xi_B(t_4) \rangle \langle m_f | m_\alpha = 0 \rangle \langle m_\alpha = 0 | m_i \rangle}{\sum_{k=-1}^1 \langle m_f | m_\alpha = k \rangle \langle m_\alpha = k | m_i \rangle}. \quad (23)$$

By imposing the condition

$$\sum_{k \neq 0} \langle m_f | m_\alpha = k \rangle \langle m_\alpha = k | m_i \rangle = 0 \quad (24)$$

we have

$$\langle \Pi_B \rangle_w = \langle \xi_f(t_f) | U(t_f, t_4) \Pi_B | \xi_B(t_4) \rangle = 1 \quad (25)$$

(where the last equality is obtained only with an ideal projector  $\Pi_B = |\xi_B(t_4)\rangle\langle\xi_B(t_4)|$ ) and also

$$\langle \Pi_{\bar{B}} \rangle_w = 0. \quad (26)$$

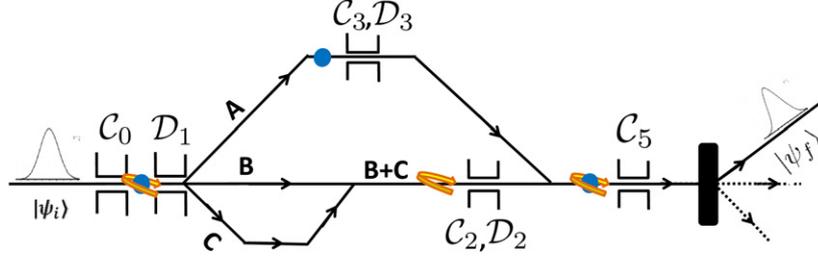
The result  $\langle \Pi_{\bar{B}} \rangle_w = 0$  can in principle be checked by recombining the paths A and C and performing a WM along that recombined path, in full analogy with the WM of  $\langle \Pi_{\bar{A}} \rangle_w$ .

The conditions to get vanishing transition elements (16) and (24) can be solved jointly. The solution (for angles  $\alpha$  and  $\phi$  coplanar with the  $z$  axis) is

$$\alpha = 2 \arccos \left( \frac{1}{2} + \frac{\sqrt{5}}{10} \right)^{1/2} \simeq 63.4^\circ \quad (27)$$

$$\phi = 2 \arccos \left( \frac{1}{2} - \frac{1}{\sqrt{5}} \right)^{1/2} \simeq 153.4^\circ. \quad (28)$$

Analogue solutions are readily obtained for other combinations (e.g, different values of  $m_i$ ,  $m_f$ ) of pre-selected and post-selected states.



**Figure 2.** The ‘Cheshire cat’ setup for spin-1 particles. The particle (pictured by a blue circle) and the spin projection  $J_\gamma$  (represented by a rotating arrow) along a specifically chosen direction  $\hat{\gamma}$  (defined in the text) appear to travel along different paths. Indeed given the pre- and post-selected states  $|\psi_i\rangle$  and  $|\psi_f\rangle$ , an apparatus positioned along A detects the particle but no spin  $J_\gamma$  whereas an apparatus along B + C detects no particle but does detect its spin (see section 4 for details).

#### 4. The Cheshire cat grin with spin-1 particles

The ‘grin without a cat’ that Alice experienced in Wonderland was introduced in the context of WM by Aharonov and Rohrlich [7]. The idea is that given pre-and post-selected states, the cat—the particle—can only be found (by performing a WM) along a given box or path, whereas the grin—a property of the particle—can only be found (by performing another WM) along a different path. The idea has been receiving increased interest recently both in refining theoretical aspects and in the form of concrete proposals [3–5]. These proposals all involve optical schemes. We give here instead a ‘Cheshire cat’ example with massive spin-1 particles that can in principle be realized in atomic interferometry experiments.

Our scheme is based on a setup almost identical to the one presented in figure 1, with the initial and final states given by equations (7) and (11) respectively. The focus here is on path A on the one hand and the recombined path B + C on the other, as portrayed in figure 2. The results given in section 3.2 hold, i.e. we choose  $\alpha$  and  $\phi$  obeying equations (16) and (17) so that  $\langle \Pi_{\hat{A}} \rangle_w = 0$ . This means that the apparatus  $\mathcal{D}_2$  and  $\mathcal{D}_3$  interacting with the wavepacket only detect the particle on path A and detect nothing on path B + C. The particle hence took path A.

Let us now introduce the spin projection  $J_\gamma$  along the axis  $\hat{\gamma}$  as the system property we wish to follow from  $t = t_i$  to  $t = t_f$  as the system goes from the initial to the final states. We choose  $\gamma$  such that

$$\langle m_f | J_\gamma | m_\alpha = 1 \rangle = 0 \quad (29)$$

giving  $\gamma$  as a function of  $\alpha$  and  $\phi$ :

$$\gamma = -2 \arctan \left( \frac{\tan \frac{\alpha}{2} + \tan \frac{\phi}{2} - \sqrt{\sec^2 \frac{\alpha}{2} \sec^2 \frac{\phi}{2} + 1}}{\tan \frac{\alpha}{2} \tan \frac{\phi}{2} - 1} \right) + 2\pi n. \quad (30)$$

(Recall that  $\alpha$  and  $\phi$  are related by equation (17) so that for a fixed post-selected state  $\gamma$  only depends on  $\alpha$ .)

Let us place an apparatus  $\mathcal{C}_0$  just after the initial state has been launched, before the SG type of device  $\mathcal{D}_1$ . The weak value  $\langle J_\gamma(t_0) \rangle_w$  measured at  $t_0 \simeq t_i$  is

$$\langle J_\gamma(t_0) \rangle_w = \frac{\langle m_f | J_\gamma | m_i \rangle}{\langle m_f | m_i \rangle} \quad (31)$$

which is typically non-zero. Let us position an apparatus  $\mathcal{C}_5$  just before post-selection takes place (see figure 2). The weak value at  $t_5 \simeq t_f$  is again given by

$$\langle J_\gamma(t_5) \rangle_w = \langle J_\gamma(t_0) \rangle_w = \frac{\langle m_f | J_\gamma | m_i \rangle}{\langle m_f | m_i \rangle} \quad (32)$$

and therefore non-zero.

Let us now place an apparatus  $\mathcal{C}_2$  localized on the recombined  $B + C$  branches (see figure 2). When interacting with the system  $\mathcal{C}_2$  measures the weak value of  $J_\gamma$  along that path, denoted  $\langle J_\gamma^{\bar{A}} \rangle_w$ . The system wavefunction is given by equation (14) and  $\mathcal{C}_2$  couples to the last term only (the term describing the path  $B + C$ ). The weak value  $\langle J_\gamma^{\bar{A}} \rangle_w \equiv \langle J_\gamma(t_2) \Pi_{\bar{A}}(t_2) \rangle_w$  is given by

$$\langle J_\gamma^{\bar{A}} \rangle_w = \frac{\langle \xi(t_f) | U(t_f, t_2) | \xi_{B+C}(t_2) \rangle \sum_{k=-1,0} \langle m_f | J_\gamma | m_\alpha = k \rangle \langle m_\alpha = k | m_i \rangle}{\langle m_f | m_i \rangle} \quad (33)$$

$$= \frac{\langle m_f | J_\gamma | m_i \rangle}{\langle m_f | m_i \rangle} \quad (34)$$

where we have used equation (29) and assumed an ideal projection  $\Pi_{\bar{A}}(t_2) \equiv |\xi_{B+C}(t_2)\rangle\langle\xi_{B+C}(t_2)|$ . Comparing equations (31), (32) and (34), it looks as if  $J_\gamma$  had traveled entirely along the path  $B + C$ . This is confirmed by positioning an apparatus  $\mathcal{C}_3$  in order to measure  $\langle J_\gamma \rangle_w$  along path  $A$  at  $t = t_3$ . The weak value  $\langle J_\gamma^A \rangle_w \equiv \langle J_\gamma(t_3) \Pi_A(t_3) \rangle_w$  is

$$\langle J_\gamma^A \rangle_w = \frac{\langle m_f | J_\gamma | m_\alpha = 1 \rangle \langle m_\alpha = 1 | m_i \rangle}{\langle m_f | m_i \rangle} = 0 \quad (35)$$

which is seen to vanish because of the condition (29) imposed on  $\gamma$ .

If we now collect the results we see that (equations (16) and (21))

$$\langle \Pi_A \rangle_w = 1, \quad \langle \Pi_{\bar{A}} \rangle_w = 0 \quad (36)$$

which can be interpreted as meaning that, given the initial and final states  $|\psi_i\rangle$  and  $|\psi_f\rangle$ , ‘the particle has traveled along path  $A$ , as it cannot be found along the other path’. We have also obtained

$$\langle J_\gamma^A \rangle_w = 0, \quad \langle J_\gamma^{\bar{A}} \rangle_w = \langle J_\gamma(t_i) \rangle_w = \langle J_\gamma(t_f) \rangle_w \quad (37)$$

which can be interpreted as meaning that ‘the property  $J_\gamma$  of the particle traveled along route  $B + C$ , as it cannot be obtained along path  $A$ ’. We have therefore realized a setup for the manifestation of a ‘Cheshire cat’ with spin-1 systems since the grin ( $J_\gamma$ ) appears to be ‘disembodied’ from the cat (the particle).

## 5. Discussion and conclusion

In this work we have given a proposal for an implementation with spin-1 particles of the three-box paradox and of a Cheshire cat type of setup. Whether the three-box problem discussed in section 3 is really constitutive of a paradox, or perhaps more strikingly, whether there is anything like ‘disembodiment’ of a property in the Cheshire cat setup presented above hinges on the status of weak measurement (WM). Indeed, WM has remained controversial since its inception; the terms of the controversy ultimately depend on the options taken on the meaning of the theoretical entities of the quantum formalism. While the discussion of these options with regard to the status of WM is beyond the scope of the present paper<sup>3</sup>, we will nevertheless make a few remarks relative to the setups discussed above.

<sup>3</sup> The topics concerning the interpretation of WM will be discussed in a forthcoming paper in relation to other works in that area.

The crucial difference between WMs and projective measurements is that the latter suppress the entangled linear superposition of system–apparatus states (as if a collapse to a single term in the pointer basis had taken place) while the former retain the full wave aspect of the quantum system. For example in dynamical systems, where the system wavefunction is characterized by a ‘sum over paths’ as prescribed by Feynman’s propagator, an array of apparati weakly interacting with the system should in principle allow us to detect the wavefunction simultaneously propagating along the available paths [19] provided the paths are sufficiently isolated from one another. The ‘strength’ of WM is to capture this wave phenomenon—too often thought of as being a computational artifact—with apparati weakly coupled to the system. Here (see figure 1) the apparati placed at  $\mathcal{D}_2$ ,  $\mathcal{D}_3$  and  $\mathcal{D}_4$  monitor the wave properties along the different paths.

The aspect of weak values as measuring the transition amplitudes—generically written  $\langle \psi_f | O | \psi_i \rangle$  in the notation of section 2.2—is well illustrated in the Cheshire cat grin scheme. The weak value (4), whose squared norm can be seen as a renormalized transition probability, vanishes if the transition is forbidden. This is the case, according to equations (36) and (37), of the transition generated by the measurement of  $J_\gamma$  along path  $A$  and of the transition generated by the position projection along the path  $B + C$ . Associating a vanishing weak value with a forbidden transition toward a final state is therefore perfectly cogent within standard quantum mechanics—provided it is kept in mind that the wavepackets take all the available paths simultaneously.

Finally, a possible tentative manner to implement experimentally the schemes developed above for spin-1 particles would be to employ atoms in setups based on well-established atomic interferometry experiments [10]. For example, based on the setup of [13] (a so-called ‘*Stern–Gerlach atom interferometer*’), hydrogen atoms can be prepared in the initial state  $2s_{1/2}$ ,  $J = 1$ ,  $m_z = 0$  (where  $J$  denotes the total angular momentum of the hyperfine Hamiltonian) and passed in a region containing a magnetic field. This yields a coherent superposition of atoms in different states  $|m_\alpha\rangle$  which is finally projected to a desired final state by using a polarizer and a time of flight detection scheme. The WMs of the projectors  $\Pi_A$ ,  $\Pi_{\bar{A}}$  etc could be realized by a selective laser excitation of a given  $|m_\alpha\rangle$  manifold. If the excitation pulse is an  $n$  photon coherent state, the measurement is weak provided  $n$  is large (the detection of the overlap between the original state  $|n\rangle$  and the  $|n - 1\rangle$  photon state after absorption of a photon gives almost no information on the path) and the transition to an excited state does not change the kinetic energy of the atomic wavepacket. We note that a closely related problem (a three-box quantum game in which individual boxes can be addressed, though the resulting wavefunctions cannot be recombined) has been very recently realized experimentally with a three-level system using the  $^{14}\text{N}$  nuclear spin ( $I = 1$ ) of the nitrogen vacancy center in diamond, the preparation and readout being performed by manipulating the  $\text{NV}^-$  electronic spin ( $S = 1$ ) with  $m_S$  and  $m_I$  selective microwave pulses [20].

As for the Cheshire cat property  $\langle J_\gamma^A \rangle_w = 0$ , it could be possible to experimentally observe this property indirectly by inducing at  $\mathcal{C}_3$  (see figure 2) a weak rotation of  $\mathbf{J}$  along the axis  $\gamma$  by a small angle  $\varepsilon_\gamma$ . If the rotation is sufficiently weak so that  $\exp(-iJ_\gamma\varepsilon_\gamma) \approx 1 - iJ_\gamma\varepsilon_\gamma$  holds (while still being detectable in the statistics of the post-selection), then a rotation along path  $A$  will not affect the state prior to post-selection, since  $\langle m_f | \exp(-iJ_\gamma\varepsilon_\gamma) | m_\alpha = 1 \rangle \approx \langle m_f | m_\alpha = 1 \rangle - i\varepsilon_\gamma \langle m_f | J_\gamma | m_\alpha = 1 \rangle$ , the last term vanishing by equation (35). Hence the rotation at  $\mathcal{C}_3$  will have no effect and will not modify the post-selection statistics, a statement that is equivalent to having a vanishing weak value  $\langle J_\gamma^A \rangle_w = 0$ . On the other hand, if the same weak rotation along  $\gamma$  generated by  $J_\gamma$  is performed at  $\mathcal{C}_0$ ,  $\mathcal{C}_2$  or  $\mathcal{C}_5$  then the post-selection statistics determined experimentally will be affected, and will be so in exactly the

same way. Thus everything happens as if  $J_\gamma$  had traveled along the route  $\mathcal{C}_0 - \mathcal{C}_2 - \mathcal{C}_5$ , but not along path A.

Summarizing, we have given a proposal implementing the three-box paradox and a Cheshire cat grin scheme for massive spin-1 systems. Besides giving a concrete rendering of paradigmatic examples of WM, employing a definite physical system sheds light on the peculiar quantum properties unraveled by WM while avoiding the ambiguities of the original ideal three-box setup that have given rise to several controversies. In principle the proposed schemes could be implemented in atomic interferometry experiments.

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