Entanglement, the quantum formalism and the classical world

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Abstract. 75 years after the term "entanglement" was coined to a peculiar feature inherent to quantum systems, the connection between quantum and classical mechanics remains an open problem. Drawing on recent results obtained in semiclassical systems, we discuss here the fate of entanglement in a closed system as Planck's constant becomes vanishingly small. In that case the generation of entanglement in a quantum system is perfectly reproduced by properly defined correlations of the corresponding classical system. We speculate on what these results could imply regarding the status of entanglement and of the ensuing quantum correlations.

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INTRODUCTION

Schrödinger coined the term "*entanglement*" in 1935 [1, 2] to describe the fact that when two systems interacted, the resulting state could "*no longer be described in the same way as before, viz. by endowing each of them with a representative* [ie vector state in Hilbert space] *of its own*. [...] *By the interaction the two representatives have become entangled*" [2]. And he added in the same first paragraph of Ref. [2] the now celebrated phrase: "*I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought*". And indeed, there is a-priori nothing like entanglement in classical mechanics. Our aim in the present note is to give some results on the properties of entanglement as the classical limit is approached and make a few remarks on the fate of entanglement in the classical world.

The understanding of entanglement has made some progress since 1935, in particular these last 20 years with the advent of quantum information related works. However these works deal essentially with qubits (a two state system for which the action – the spin – is of the order of $\hbar$) relevant to investigate only some of the conceptual aspects (quantum logic, separability, communication constraints) that lie at the root of entanglement. Other aspects, dealing with the quantum-classical correspondence and the emergence of classical mechanics, call for additional tools. These are to be found in semiclassical physics.

In short semiclassical physics [4, 3] investigates the properties of quantum systems by making an explicit link with the properties of the corresponding classical system. In general, the existence of a corresponding classical system is guaranteed by canonical quantization (the classical and quantum Hamiltonians have the same functional dependence on phase-space variables and operators respectively), and the quantum system behaves semiclassically provided the actions S_i of the system are huge relative to Planck's constant, ie $S_i/\hbar \to \infty$, or in short $\hbar \to 0$. In other cases the corresponding classical system is obtained from the first order expansion of the path integral form of the evolution operator. In both situations the validity of the quantum-classical correspondence hinges on the fact that as $\hbar \to 0$ the wavefunction propagates (simultaneously) along all the available trajectories of the corresponding classical system.

Given that entanglement is a characteristic quantum property, it could appear at first sight far-fetched to look for any quantum-classical correspondence in entangled semiclassical systems. This is why we will first give a brief overview of the vast amount of works that have studied the effect of the classical underlying dynamics on the entanglement evolution. Then we will summarize some results concerning entanglement generation that are obtained for a scaling system whose dynamics is invariant as $\hbar \to 0$, giving rise to an apparent paradox: entanglement increases as the classical limit is approached, but the amount of entanglement is captured with increasing accuracy by probabilities obtained from the corresponding classical system. We will close with some remarks regarding entanglement in the classical world, the role of decoherence, and the status of the present quantum mechanical formalism.

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ENTANGLEMENT IN SEMICLASSICAL SYSTEMS

Assume two particles, each endowed with its own dynamics obtained from a single particle Hamiltonian H_i with $i = 1,2$ become coupled at some time $t = 0$ via an inter-particle potential term V_{12}. Classically, the equations of motion are obtained from the total Hamiltonian

$$H = H_1 + H_2 + V_{12}. \tag{1}$$

The individual particle trajectories are coupled by the V_{12} term, and if one resorts to a statistical description, the typical distributions that can be defined for the entire system are given by summing correlated single particle distributions.

Moving to the quantum case, assume each single particle system described by (the now quantized) H_i is in the semiclassical regime, ie the wavefunction propagates along the classical trajectories of the classical single particle Hamiltonian; as a result the dynamical and statistical properties of the quantum system can be obtained from the properties of the classical trajectories, in particular from the classical periodic orbits (see Refs. [3, 4] and Sec. 3 of [5] for a short exposition). When the systems are coupled, the total system wavefunction $|\psi(t)\rangle$ is built from the eigenstates of the Hamiltonian (1); it is expressed over the product Hilbert space basis $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$. The change brought by the inter-particle interaction in the dynamics of the individual particles can be followed by computing the reduced density matrices. Hence for example from the total density matrix

$$\rho(t) = |\psi(t)\rangle \langle\psi(t)| \tag{2}$$

the reduced density matrix $\rho_1(t)$ giving particle 1's properties is obtained by averaging over particle 2's possible outcomes

$$\rho_1(t) = \mathrm{Tr}_2 \rho(t). \tag{3}$$

Generically $|\psi(t)\rangle$ will be entangled. Even if initially $|\psi(t=0)\rangle$ is chosen as a product state, entanglement will build up during the ensuing unitary evolution. To quantify entanglement it is customary to employ the linear entropy

$$\Omega(t) = 1 - \mathrm{Tr}_1 \rho_1^2(t) \tag{4}$$

which is easier to compute than the Von Neumann entropy. $\Omega(t)$ vanishes for product states and takes its highest value for maximally entangled states. Note that Ω is symmetric, ie $\mathrm{Tr}_1\rho_1^2 = \mathrm{Tr}_2\rho_2^2$.

Given our original assumption regarding the semiclassical regime, in the uncoupled case $\rho_i(t)$ is given by a sum of classical amplitudes. It makes sense to expect that the $\rho_i(t)$ still follow to some extent the quantum-classical correspondence even in the presence of V_{12} (and especially so if the coupling is weak). Then we see from Eq. (4) that the quantum-classical correspondence will transpire in the generation of entanglement. This observation prompted several works involving mostly numerical and sometimes analytical approaches (it is impossible to cite all these works here; see eg [6, 7, 8, 9, 10, 11, 12, 13] and Refs. therein). One of the main issues concerns the relationship between classical chaos and entanglement: grounded on the general idea that classically chaos enhances the diffusion in phase space, it seems natural to expect that quantum systems with a classically chaotic counterpart will entangle more efficiently than those having a classical counterpart displaying regular dynamics.

Things were not so simple however (counterexamples were readily obtained). Briefly stated, what matters is that the classical distributions corresponding to the initial uncoupled quantum states dynamically evolve so as to mix significantly. Then in the quantum system this will correspond to mixtures of the probability amplitudes, yielding a non-diagonal density matrix. The type of classical flow leading to such a situation does not need to depend on the dynamical regimes of the uncoupled systems, but rather on the coupling parameters and on the choice of initial distributions. This is why it was suggested [14] that a relevant comparison of entanglement generation with the underlying classical dynamics should be carried out in systems in which the coupling interaction V_{12} that generates the entanglement is the one that drives the classical dynamical regime (ie that creates chaos). Particularly interesting systems are those in which V_{12} depends on a parameter k that can be varied so that for certain values of k the classical dynamics is regular, but as k is varied the dynamics becomes of the mixed phase-space type, or chaotic. An illustration concerning such a system is given in Fig. 1.

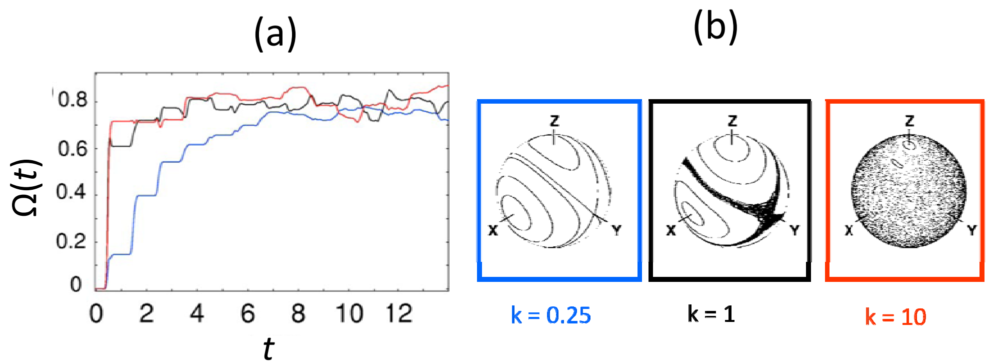

FIGURE 1. (a) The linear entropy $\Omega(t)$ gives the entanglement generation rate for a bipartite system uncoupled at $t = 0$ (t is given in terms of the number of times the particles interact through V_{12}). Each curve corresponds to a different value of k, characterizing the strength of the contact interaction V_{12} . (b) The surface of section for the corresponding classical system is shown for different values of k. For $k = 0.25$ the classical system displays regular dynamics (the entanglement rate in the corresponding quantum system is shown in blue in (a)). For $k = 1$ ($k = 10$) the system has mixed phase-space (chaotic) dynamics and the corresponding curves are shown in black (red) in (a).

ENTANGLEMENT AS $\hbar \to 0$

Entanglement and classical distributions

From the example shown in Fig. 1, it can be remarked that regular or chaotic dynamics in the classical system lead to a comparable entanglement rate. Generically however, it is true that chaos tends to translate into more entanglement in arbitrary situations. It is nevertheless difficult to make universal statements: since there is no classical quantity corresponding to entanglement, it is not possible to compute a well-defined classical version of Eq. (4). Moreover for typical systems that have been investigated numerically, $\hbar$ is still far from being negligible (relative to the actions of the systems), since one must keep in mind that the size of the Hilbert space (and hence of the quantum computations) increase with the actions and the computations become therefore untractable.

Except maybe in a weak coupling regime, where exact semiclassical expansions can be obtained analytically it is hardly possible in definite systems to explain the behaviour of entanglement in terms of the dynamics of classical distributions. Only very general arguments can be given, for example when the corresponding classical dynamics is regular the entanglement behaviour is seen to display regular oscillations (due to coherent revivals of the quantum states constrained to remain in regular structures – the torii), whereas this is not the case in the presence of classically chaotic dynamics. In principle a purely formal equivalent of Eq. (4) can be defined [9] by replacing ρ_1 or ρ_2 by classical distributions and the trace by phase-space integrals. But the linear entropy thus defined is not symmetric (ie, the quantum relation $\mathrm{Tr}_1\rho_1^2 = \mathrm{Tr}_2\rho_2^2$ is not necessarily verified with phase-space integrals) and hardly has a physical meaning within classical mechanics. Our strategy that we pursue below is to employ a quantum system in which the generation of entanglement is due to a contact interaction producing in the classical counterpart identified changes in the motions in each of the particles. A nice property of the system is that it scales with $\hbar$, meaning that the dynamics stays *constant* as $\hbar$ is decreased. This allows to effectively investigate entanglement for a given dynamics as $\hbar \to 0$.

Scaling system

In a classical system the action S is the quantity having the dimension of $\hbar$. In the semiclassical approximation the quantum wavefunctions take the generic form

$$\psi \sim \sum_t \left| \det \frac{-\partial^2 S(\mathbf{q}_t)}{\partial q^i \partial q_0^j} \right|^{1/2} \exp iS(\mathbf{q}_t)/\hbar \tag{5}$$

for a single particle, where t runs over the classical trajectories reaching $\mathbf{q}$ from the initial point $\mathbf{q}_0$. In general, if one increases S so that $\hbar/S \to 0$, the dynamics (both classical and quantum) is modified. For a two-particle system the action is separable only at $t = 0$ but can often be written as $S = \sum_i S_i + \sum_{ij} S_{ij}$ (where the indices run on the particles and/or on the degrees of freedom). Now as $\hbar/S \to 0$ the dynamics as well as the entanglement properties in the quantum system will be modified. If the system scales however, we can modifiy the parameters and dynamical variables of the system so that $S_i \to \tilde{S}_i/\kappa$, $S_{ij} \to \tilde{S}_{ij}/\kappa$ (and thus $S \to \tilde{S}/\kappa$) and $q^i \to q^i \kappa^\gamma$ where κ is a constant. As can be seen from Eq. (5) this scaling is tantamount to keeping the action and the dynamics constant but rescaling an effective Planck's constant defined by $\hbar_{\text{eff}} = \kappa\hbar$, so that by choosing smaller values of κ one can effectively investigate entanglement as $\hbar \to 0$.

We have recently investigated entanglement as $\hbar \to 0$ in a two-particle scaling system [15]. In short, bipartite entanglement is generated by repeated inelastic scattering of two particles – a light structureless particle and a heavy rotating particle, modeled by a symmetric top with angular momentum N and energy $E_N \propto N(N+1)$. The scattering potential V_{12} is taken to be a contact interaction so that the light incoming particle receives a kick when it hits the rotating top. To account for repeated scattering we add an attractive field between both particles. Labelling $|F_N\rangle$ and $|N\rangle$ the quantum states of the light and heavy particles respectively ($|F\rangle$ depends implicitly on N because the total energy and the total angular momentum of the entire system are conserved) a typical quantum state takes the form

$$|\psi\rangle = \sum_N B_N |F_N\rangle |N\rangle, \tag{6}$$

showing entanglement between the rotational state of the symmetric top and and the energy and angular momentum of the light particle (the B_N are just coefficients depending on the scattering matrix elements). Initially both particles are uncoupled, the state being $|F_0\rangle |N_0\rangle$.

The classical version of the system hinges on employing the semiclassical link between the deflection angle ϕ produced on the motion of the light particle by the kick and the quantum scattering matrix. According to this link, the kick strength can be parameterized by a coupling strength k, each value of k corresponding to a different quantum scattering matrix. The scaling involves the angular momenta (which are actions) of the light and heavy particles and the radial action of the light particle. The dynamics remains invariant provided the orbital period of the light particle and the rotational period of the heavy particle are adjusted accordingly.

Entanglement and classical probabilities

A first result [15] concerns a simple scaling formula for the linear entropy (4) quantifying entanglement. Assume two values of the effective Planck constant with $\tilde{\hbar}_{\text{eff}} < \hbar_{\text{eff}}$. Then

$$\tilde{\Omega}(t) = 1 - \frac{\tilde{\hbar}_{\text{eff}}}{\hbar_{\text{eff}}}(1 - \Omega(t)), \tag{7}$$

so that entanglement increases as $\hbar_{\text{eff}} \to 0$. This is to be expected since the size of the Hilbert space increases with decreasing $\hbar_{\text{eff}}$.

The second more surprising result was to show that the linear entropy can be given to a good approximation by computing the weights of the evolving classical distributions. Indeed, the initial quantum density matrix $|F_0\rangle |N_0\rangle \langle F_0| \langle N_0|$ has a straightforward classical counterpart (a classical distribution). This distribution evolves encompassing several values of the classical angular momentum N. If one divides classical phase-space into q cells, each cell corresponding to the volume occupied by a quantum state $|N\rangle \langle N|$ (q being the total number of quantum states), then the classical distribution spreads across a certain number of such cells. By simply counting the relative fraction of the classical distribution in each cell, we define the probabilities $p_N^{cl}(t)$, corresponding to the probability of having the classical system in which the top has an angular momentum $N \pm \Delta_N/2$ (and the light particle the relevant angular momentum and energy as imposed by the conservation laws). Δ_N is a measure of the width of the classical cell.

The probabilities $p_N^{cl}(t)$ are asymptotically close to the weights of the reduced density matrices, ie letting ρ_1 of Eq. (3) refer to the reduced density matrix of the rotating top, then

$$\rho_1(t) = \sum_N p_N(t) |N\rangle \langle N| \tag{8}$$

with $p_N(t) \approx p_N^{cl}(t)$. This is due to the fact that the off-diagonal elements of the quantum scattering matrix oscillate wildly as $\hbar \to 0$, so that the interference terms resulting from the incoherent sum of thousands of terms tends to vanish.

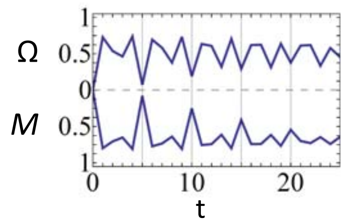

FIGURE 2. The top panel shows the entanglement rate for the bipartite quantum system (the time is given by the number of contact interactions between the two particles). The bottom panel shows the evolution of the mutual information (regarding the value of N) of the corresponding classical system. M is obtained from the evolution of classical statistical distributions.

Classically the quantity

$$M(t) = 1 - \sum_N \left[p_N^{cl}(t) \right]^2 \tag{9}$$

is the mutual information quantifying the amount of mixing among the different cells of the classical system, each cell being defined by the symmetric top having a mean rotation number N and the light particle having the corresponding mean energy. An illustration of this behaviour is given in Fig. 2.

ENTANGLEMENT, THE QUANTUM FORMALISM AND THE CLASSICAL WORLD

Entanglement *effectively* disappears in the classical limit

The apparent paradox is that as the classical limit is approached, entanglement increases but can be obtained from classical quantities. In other words, the quantum system has a total non-diagonal density matrix readily obtained from Eq. (6), but looking at a single subsystem (whose properties are coined in its reduced density matrix) the situation is identical to the one that would follow if the total system was given by a diagonal density matrix

$$\rho^{cl}(t) = \sum_N p_N^{cl}(t) \left| F_N \right\rangle \left\langle F_N \right| \otimes \left| N \right\rangle \left\langle N \right| . \tag{10}$$

Here the cl superscript refers both to classical *correlations* (ρ^{cl} is a convex combination of orthogonal projectors) and to classical *dynamics* (the p_N^{cl} are probabilities obtained from the classical system). In principle, the only way to distinguish ρ from ρ^{cl} would involve measuring two-particle observables. But in practice, as $\hbar \to 0$ the coherences (in the "pointer basis") of typical two-particle observables would yield interference patterns with vanishing (and therefore undetectable) wavelengths, at any rate smaller than the size of an elementary particle [16].

The upshot is that at the end of the day, entanglement persists in the classical world at a formal level but is devoid of any physical meaning: the quantum system *effectively* evolves according to Eq. (10), that is as a classical system. The situation is similar to the one encountered in environmental decoherence, where it can be seen that by coupling an entangled system to an environment, the reduced density matrix of the system ρ_S obtained by averaging over the environment states behaves classically (in the sense of correlations) for all practical purposes. This is so because the non-diagonal terms of ρ_S are strongly suppressed while the total density matrix accounting for the coupled system and environment remains entangled. What we have shown here is that for semiclassical systems there is no need to introduce an environment: the individual components of the entangled system behave effectively according to the laws of classical mechanics.

The point that remains to be discussed is to what extent an *effective* solution ('for all practical purposes') can be claimed to be a satisfactory solution. Indeed, the system appears to behave classically, though formally it remains entangled (and all the more so as the classical limit is approached). It is well-known [17, 18] that environmental decoherence is not a real solution that accounts for the appearance of the classical world, unless one assumes there was no problem to begin with (and unless one discards multiple universe interpretations, which has other problems on its own). The situation is similar here: there is no problem in accounting for the appearance of a classical behaviour for entangled semiclassical systems in the classical limit provided one assumes there was no problem to begin with. The phrase 'no problem to begin with' is connected to the status given to the quantum formalism with regard to reality.

Reality and the quantum formalism

Crudely speaking, there are two kinds of terms in our physical theories: some theoretical terms refer to something “out there” in the Universe (they are ontological, or referring terms) while other terms are purely knowledge related (epistemic terms) [19]. If we take classical mechanics as a paradigm, we encounter referring terms: the velocity, the position, the applied forces refer to particles and fields that, paraphrasing Popper, we can kick and that can kick us back [20]. On the other hand classical mechanics also contains epistemic, non-referring terms: the Lagrangian or the action encode the entire dynamical information of a given system. They live in an abstract multi-dimensional configuration space and we can certainly not kick them. The determination of which (if any) theoretical terms ascribe reference is not an arbitrary choice that we can freely make (this is what does not allow the Bohmian model to be considered as a realist account of quantum phenomena [5]. We need observational warrants in order to capture the ontological features of a theory, and these only emerge by combining different types of experiments, observations and logical inference.

The standard formalism of quantum mechanics does not allow to ascribe reference in an undisputed and unambiguous manner. It is impossible to propose a testable ontology out of the formalism as it stands today, and only weak statements about what the theoretical terms could refer to can be made (like the eigenvalue-eigenstate link, or the existence of invariant quantities like the mass, the charge...). The only consistent interpretation taking the theoretical terms of the standard formalism (like the wavefunction) at face value as exisiting in reality would need to rely on some form of the many worlds interpretation.

In this context, a prudent attitude would consist in endorsing the epistemic view. If we assume the wavefunction is an epistemic term encoding information about the system then we do not need to commit ourselves to solutions involving objective processes that would distinguish two density matrices that are different in nature (ρ is entangled while ρ^{cl} is not diagonal in the "pointer" basis) but nevertheless give exactly the same predictions (this, for all practical purposes, impossible by definition!). In short if ρ appears as formally entangled but in the classical limit the entanglement cannot be detected, then from an epistemic point of view it is correct to claim that entanglement has vanished. This viewpoint is particularly consistent from a semiclassical perspective [5] because the basic quantum theoretical entity (the wavefunction) appears as being built from classical quantities having a non-referring, *epistemic* status (as is obvious from Eq. (5)).

There is no need to say that this situation is hardly satisfactory. But it leads us to speculate on whether the problems regarding the meaning and the interpretation of the quantum mechanical features, such as entanglement, and their behaviour when studying the quantum-classical transition, is not to be found in trying to ascribe reference to the theoretical terms of the standard formalism. At the classical level it would not make sense to build the ontology for classical mechanics out of the Hamilton-Jacobi formalism. And the semiclassical approach shows that by construction the quantum mechanical quantities tend when $\hbar \to 0$ to represent classical statistical distributions expressed in the Hamilton-Jacobi formalism. While it is true that logically nothing impedes that an apparently non-referring formalism in one theory ends up referring to something real (and there are historical examples that could support this assertion), it is noteworthy that 75 years after the advent of entanglement, its nature still remains elusive.

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